SEMI-RIEMANNIAN GEOMETRY

Definition 1:

A bilinear form $\beta : V \times V \to \mathbb{F}$ on an $\mathbb{F}$-vector space V is called non-degenerator if the maps $B_R : V \to V^*$ given by $B_R : v \mapsto \beta(v, \cdot)$ and $B_L : V \to V^*$ given by $B_L : v \mapsto \beta(\cdot, v)$ are both linear isomorphism.

If V is finite dimensional, then B_R is an isomorphism iff B_L is an isomorphism and then β is non-degenerate iff $\beta(v, \omega) = 0 \ \forall \ \omega \in V$, then $v = 0$.

Definition 2:

A (real) scalar product on a (real) finite dimensional vector space V is a non-degenerate bilinear form $\beta : V \times V \to \mathbb{R}$.

A (real) scalar product space is a pair (V, β) where V is a real vector space and β is a real scalar product.

Definition 3:

Let V be a complex vector space. An $\mathbb{R}$-linear map $\beta : V \times V \to \mathbb{C}$ that satisfies $\beta(\alpha v, \omega) = \bar{\alpha}\beta(v, \omega)$ and $\beta(v, \alpha\omega) = \alpha\beta(v, \omega)$ for all $\alpha \in \mathbb{C}$ and $v, \omega \in V$ is called a sesquilinear form. If additionally $\beta(v, \omega) = \overline{\beta(v, \omega)}$ then β is called a Hermitian form.

If a Hermitian form is non-degenerate, we say that ~~f~~ it is a ~~Hermil~~ Hermitian scalar product. Then (V, β) is a Hermitian scalar product space.

Definition 4:

Let β be a (real) scalar product or Hermitian scalar product on a vector space V. Then

1. β is +ve (resp. −ve) definite if $\beta(v, v) \geq 0$ (resp. $\beta(v, v) \leq 0$) for all $v \in V$, and $\beta(v, v) = 0 \Rightarrow v = 0$.
2. β is +ve (res. −ve) semi definite if $\beta(v, v) > 0$ (resp. $\beta(v, v) < 0$) for all $v \in V$.

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An inner product is a +ve definite scalar product (or +ve definite Harmitian scalar product in Complex case).

Definition 5:

Let $(V, \langle \cdot, \cdot \rangle)$ be a scalar product space. A non-zero vector $v \in V$ is called

1. space like if $\langle v, v \rangle > 0$.
2. light like or null if $\langle v, v \rangle = 0$
3. time like if $\langle v, v \rangle < 0$
4. Non null if V is either time like or space like.

The terms space like, null (light like) and time like indicate the causal charecter of a vector.

If (M, g) is a semi-riemannian manifold, if each tangent space is a scalar product space and the above defn applies, we define $\|v\| = |\langle v, v \rangle|^{1/2}$ which we call the length of v.

Definition 6:

The set of all null vectors in a scalar product space is called the null cone or light cone.

If (M, g) is a semi-Riemannian manifold, then the null cone in T_pM is called the null cone at p.

Definition 7:

Let $I \subseteq \mathbb{R}$ be an interval. A curve $\gamma : I \to (M, g)$ is called space like, null, time like or non null according as $\gamma(t) \in T_{\gamma(t)} M$ is space like, null, time like or non null for all $t \in I$.

Definition 8:

A curve $\gamma : I \to (M, g)$ called causal if $\gamma(t) \in T_{\gamma(t)} M$ is either time like or null for all $t \in I$.

Definition 9 :

Let (M, g) be a semi Riemannian manifold. If $\gamma : [a,b] \to M$ is piecewise smooth curve, then

$$L_{\gamma(a),\gamma(b)}(\gamma) = \int_a^b \left|\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle\right|^{1/2} dt$$

is called the arc length or simply the length of the curve.

Definition 10 :

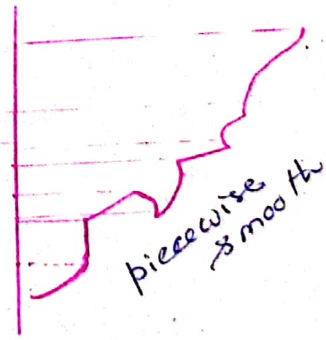
A +ve reparametrization of a smooth curve $\gamma : [a,b] \to M$ is a curve defined by composition $\gamma \circ h : [a', b'] \to M$, where $h : [a', b'] \to [a,b]$ is a smooth monotonically increasing bijection.

Similarly, a -ve reparametrization is give by composition with a smooth monotonically decreasing bijection $h : [a', b'] \to [a,b]$.

By a parametrization we shall mean either +ve or a -ve reparametrization.

Remark 1

Suppose that $\gamma : [a,b] \to M$ be a piecewise smooth curve that is a continuous curve such that for some partition $a = t_0 < t_1 < \dots < t_n = b$, γ is smooth on each $[t_{i-1}, t_i]$, $1 \leq i \leq n$.



A +ve reparametrization of γ is a curve $\gamma \circ h : [a', b'] \to M$ where $h : [a', b'] \to [a,b]$ is a monotonically increasing continuous bijection that is smooth on each interval $h^{-1}([t_{i-1}, t_i])$; $1 \leq i \leq n$.

Negative reparametrization can be seen analogously.

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Remark 2

The integrals above are well-defined, since $\dot{\gamma}(t)$ is defined and continuous expect for a finite no. of points in $[a,b]$. Also, it is important to note that by standard change of variable arguments, a reparametrization $\sigma = \gamma \circ h$ does not change the arc length of the curve, i.e.,

$$\int_a^b |\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle|^{1/2} \, dt = \int_{h^{-1}(a)}^{h^{-1}(b)} |\langle \dot{\sigma}(u), \dot{\sigma}(u) \rangle|^{1/2} \, du.$$

Thus the arc length of a piecewise smooth curve is a geometric property of the curve, i.e., a semi Riemannian ivariant.

Definition 11 :

Let (M, g) be a semi-Riemannian manifold. Let $I \subseteq \mathbb{R}$ be an interval (possibly infinite). If $\gamma : I \to M$ is a smooth curve with $\|\dot{\gamma}(t)\| = 1$, $\forall t \in I$, then we say that γ is a unit speed curve.

If $\gamma : I \to M$ is a curve such that $\|\dot{\gamma}(t)\|$ is never zero, then choosing a reference ~~tee~~ $t_0 \in I$, we may define an arc length function $l : I \to I' \subseteq \mathbb{R}$ by

$$l(t) = \int_{t_0} |\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle|^{1/2} \, dt.$$

For a finite interval $[a,b]$ the reference t_0 is most often taken to be the left end point a. Since $\frac{dl}{dt} = |\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle|^{1/2} > 0$, we may invert to find l^{-1}

and then reparametrized

$$\sigma(s) = \gamma(L^{-1}(s)).$$

so, it is than easy to see from chain rule that the resulting curve is a unit speed curve.

Conversely, if γ is a unit speed curve, then the arc length from $\gamma(s_1)$ to $\gamma(s_2)$ is $s_2 - s_1$.

In the case of time like curves, we sometimes use τ instead of s and we refer to is as a proper time.

* Let (M, g) be a semi Riemannian manifold, suppose that $\gamma : I \to M$ is a smooth curve, i.e. self-parallel, (Auto parallel), in the sense that

$$\nabla_{\partial_t} \dot{\gamma} = 0, \text{ along } \gamma.$$

we call γ a geodesic.

If $\gamma : [a, b] \to M$ is a curve which is the restriction of a geodesic defined and an open interval containing $[a, b]$, then we call γ a (parametrised) closed geodesic segment. If $\gamma : [a, \infty) \to M$ (resp. $\gamma : (-\infty, a] \to M$) is the restriction of a geodesic, then we call γ a +ve (resp. -ve) geodesic ray.

* If the domain of geodesic is $\mathbb{R}$, then we call γ a complete geodesic. If M is an n-dimensional manifold and the image of a geodesic γ is contained in the domain of some chart with co-ordinate function $x^1, \ldots, x^n$, then the condition for γ to be a geodesic is

$$\frac{d^2 x^i \circ \gamma}{dt^2}(t) + \Gamma^i_{jk}(\gamma(t)) \frac{dx^j \circ \gamma}{dt}(t) \frac{dx^k \circ \gamma}{dt}(t) = 0, \quad t \in I, \quad 1 \le i \le n.$$

This eqn is often abbreviated

$$\frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0, \quad t \in I, \quad 1 \le i \le n.$$

These are the local geodesic eqns.

Now, consider a smooth curve γ whose image is not necessarily contained in the domain of a chart. For every $t_0 \in I$, there is $\varepsilon > 0$ such that $\gamma|_{(t_0-\varepsilon,\, t_0+\varepsilon)}$ is contained in the domain of a chart, and thus it is not hard to see that γ is a geodesic iff each such restriction satisfies the corresponding local geodesic eqⁿs. For each chart which meets the image of γ. We can convert the local geodesic eqⁿ into a system of $2n$ 1st order eqⁿ.

We let v denote a new independent variable and then we get

$$\frac{dx^i}{dt} = v^i \;; \quad 1 \le i \le n$$

$$\frac{dv^i}{dt} + \Gamma^i_{jk} v^j v^k = 0 \;; \quad 1 \le i \le n.$$

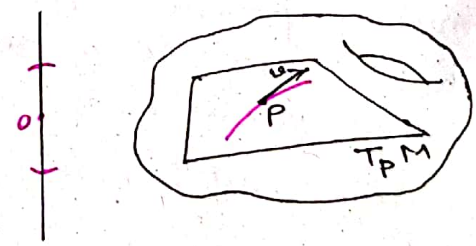
We can think of x^i and v^i as co-ordinates on TM. Once we do this we recognise that the 1st order system above is the local expression of the eqⁿs for the integral curves of a vector field on TM.

Lemma 1

For each $v \in T_pM$, there is an open interval I containing 0 and a unique geodesic $\gamma : I \to M$ such that $\dot\gamma(0) = v$. (Hence $\gamma(0) = p$).

proof:-

This follows from the standard existance and uniqueness results for differential eqⁿ.



Lemma 2

Let γ_1 and γ_2 be geodesics from $I \to M$. If $\dot\gamma_1(t_0) = \dot\gamma_2(t_0)$ for some $t_0 \in I$, then $\gamma_1 = \gamma_2$.

proof:-

If this is not true, then there must be $t' \in I$ such that $\gamma_1(t') \neq \gamma_2(t')$. Let us assume that $t' > t_0$ without loss of generality. Let $A = \{t \in I : t > 0 \text{ and } \gamma_1(t) \neq \gamma_2(t)\}$. Then A is non empty and bounded below by t_0

So, A has a infimum, say $b = \inf A$. Then definately $b \geq t_0$.

We claim that $\gamma_1(b) = \gamma_2(b)$, ~~indeed~~ indeed if $b = t_0$, then there is nothing to prove. If $b > t_0$ then $\dot{\gamma}_1(t) = \dot{\gamma}_2(t)$ on the interval (t_0, b). By continuity we have $\dot{\gamma}_1(b) = \dot{\gamma}_2(b)$.

Now the maps $t \longmapsto \gamma_1(b+t)$ and $t \longmapsto \gamma_2(b+t)$ are clearly geodesic with initial velocity $\dot{\gamma}_1(b) = \dot{\gamma}_2(b)$. By the previous lemma, $\gamma_1 = \gamma_2$ for some open interval containing b. But this contradicts the def^n of b as inf of A.

Here our assumption is not true, i.e. $\gamma_1(t) = \gamma_2(t)$, ~~$\gamma_1 = \gamma_2$~~ $\forall t \in I$.

Hence $\gamma_1 = \gamma_2$.

* A geodesic $\gamma : I \to M$ is called maximal if there is no other geodesic with open interval domain J strictly containing I that agrees with γ on I.

$$\gamma' : J \to M$$

$$\gamma'(t) = \gamma(t) \, ; \forall t \in I.$$

THEOREM 1

For any $v \in TM$, there is a unique maximal geodesic γ_v with $\dot{\gamma}_v(0) = v$.

proof :-

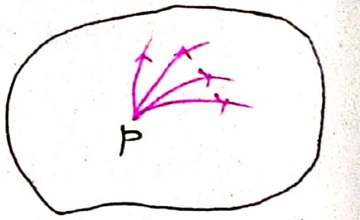
Let us take the class $\mathcal{G}_v$ of all geodesics with initial velocity v. Clearly $\mathcal{G}_v \neq \phi$ (By lemma 1). If $\alpha, \beta \in \mathcal{G}_v$ and the respective domains I_α and I_β have non-empty intersection then α, β agree on this intersection (By lemma 2).

From this we see that the geodesics in $\mathcal{G}_v$ fit together to form a manifestly maximal geodesic with domain $I = \bigcup_{\gamma \in \mathcal{G}_v} I_\gamma$.

Obiviously this geodesic has initial velocity v.

Definition 12 :

If the domain of every maximal geodesic emanating from a point $p \in T_pM$ is all of $\mathbb{R}$, then we say that M is geodesically complete at p. And a semi-Riemannian manifold is said to be geodesically complete iff it is geodesically complete at each of its points.



* Lorentzian manifold (Semi-Riemannian manifold).

Definition 13 :

A continuous curve $\gamma : [a,b] \to M$ is said to be a broken geodesic segment if it is piecewise smooth curve whose smooth segments are geodesic segment.

If t_* is a point in $[a,b]$ where γ is not smooth, then $\gamma(t_*)$ is called a broken point.

THEOREM 2

A semi-Riemannian manifold is connected iff every pair of its points can be joined by a broken geodesic $\gamma : [a,b] \to M$.

** Let $\tilde{D}_p = \{ v \in T_pM$: the geodesic γ_v is defined at least on the interval $[0,1] \}$

Then the exponential map, $\exp : \tilde{D}_p \to M$ is defined by $\exp_p(v) = \gamma_v(1)$.

Lemma 3

If γ_v is the maximal geodesic with $\gamma_v'(0) = v \in T_pM$ then for any $c, t \in \mathbb{R}$, we have $\gamma_{cv}(t)$ is defined iff $\gamma_v(ct)$ is defined. ~~We~~ When either side is defined we have $\gamma_{cv}(t) = \gamma_v(ct)$.

proof :-

Let $J_{v,c}$ be the maximal interval for which $\gamma_v(ct)$ is defined for all $t \in J_{v,c}$.

clearly, $0 \in J_{v,c}$.

Then the assignment $t \longmapsto \gamma_v(ct)$ is a geodesic with initial velocity cv. But then by uniqueness and maximality of γ_{cv}, the interval $J_{v,c}$ must be contained in the domain of γ_{cv} and for $t \in J_{v,c}$, we must have

$$\gamma_{cv}(t) = \gamma_v(ct).$$

In other words, if the right hand side is defined, then so the left and we have the equality. Now let $u = cv$, $s = ct$ and $b = 1/c$. Then we have

$$\gamma_{bu}(s) = \gamma_u(bs).$$

where if the right hand side is defined, then so the left. But this is just,

$$\gamma_v(ct) = \gamma_{cv}(t).$$

So, left and right have ~~save~~ reversed and we conclude that if either side is defined, then so the other.

Corollary

If γ_v is the maximal geodesic with $\dot{\gamma}_v(0) = v \in T_pM$, then, 1) t is ~~the~~ in the domain of γ_v iff tv is in the domain of $\exp_p$.

2) $\gamma_v(t) = \exp_p(tv)$ for all t in the domain of γ_v.

$$\left[\exp_p(v) = \gamma_v(1) \;, \therefore \exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t)\right]$$

Remark 3

The maximum geodesic through p with initial velocity v can always be written in the form $t \mapsto \exp_p(tv)$. Straight lines through $O_p \in T_pM$ are mapped by $\exp_p$ onto geodesics which are sometimes refer to as radial geodesics through p. Similarly we have radial geodesic segments and radial geodesic rays emanating from p.

proposition

$$\langle \dot{\gamma}_v(t), \dot{\gamma}_v(t) \rangle = \langle v, v \rangle \text{ for all } t \text{ in a domain of } \gamma_v.$$

THEOREM 3

Let (M, g) be a semi-Riemannian manifold and $p \in M$. Then $\exists$ an open nbd $\tilde{U}_p \subseteq \tilde{D}_p$ containing O_p s.t. ~~exp~~ $\exp_p|_{\tilde{U}_p}$ is a diffeomorphism onto its image. U_p.

proof :-

The tangent space T_pM is a vector space, which is isomorphic to $\mathbb{R}^n$ and so has a standard differential structure.

Using the result about smooth dependence on initial conditions for differential equations, we can see that $\exp_p$ is well defined and smooth in some nbd of $O_p \in T_pM$.

Let us consider the tangent map,

$$T\exp_p : T_{O_p}(T_pM) \to T_pM.$$

Let $v_{O_p} \in T_{O_p}(T_pM)$ be the velocity to the curve $t \mapsto tv$ in T_pM.

Then we have $T\exp_p(v_{O_p}) = \frac{d}{dt}\Big|_0 \exp_p(tv) = v$.

Thus $T\exp_p$ is just the canonical map $v_{O_p} \mapsto v$.

Thus $T\exp_p$ is an isomorphism. Then by inverse mapping theorem there exists an open nbd $\tilde{U}_p \subseteq \tilde{D}_p$ containing O_p such that ~~exp~~ $\exp_p|_{\tilde{U}_p}$ is diffeomorphism onto its image, i.e. $\exp_p(\tilde{U}_p) = U_p$.

Definition 14 :

A subset C of a vector space V that contains 0 is called star-shaped about 0 if whenever $v \in C$, $tv \in C$ for all $t \in [0,1]$.

Definition 15 :

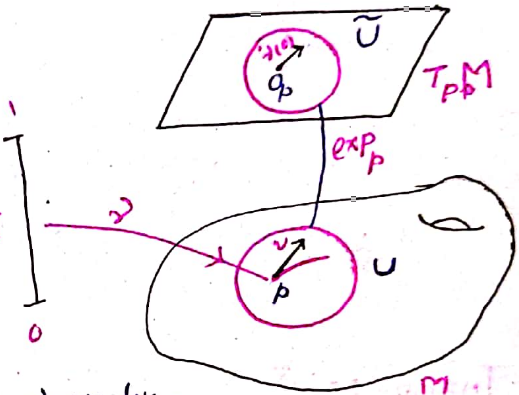
If $\tilde{U} \subseteq \tilde{D}_p$ is a star-shaped open set about $0 \in T_pM$ s.t. $\exp_p|_{\tilde{U}}$ is a diffeomorphism from $\tilde{U}_p$ onto $\exp_p(\tilde{U}) = U$, then U is called a normal nbd of p.

In this case U is also referred to as star-shaped.

THEOREM 4

If $U \subseteq M$ is normal nbd of p with corresponding pre-image $\tilde{U} \subseteq T_pM$, then for any $q \in U$ there is a unique geodesic $\gamma : [0,1] \longrightarrow U \subseteq M$ such that $\gamma(0) = p$, $\gamma(1) = q$ and $\dot{\gamma}(0) = v \in \tilde{U}$ and $\exp_p(\dot{\gamma}(0)) = \exp_p(v) = q$.

proof :-



The preimage $\tilde{U}$ corresponds diffeomorphically to U under the map $\exp_p$. Let $q \in U$ and $v = \exp_p|_{\tilde{U}}^{-1}(q)$. So that $v \in \tilde{U}$

By assumption $\tilde{U}$ is star shaped and so, the map $\rho : [0,1] \longrightarrow T_pM$, given by $t \longmapsto tv$ has image in $\tilde{U}$. But then the geodesic segment $\gamma : t \longmapsto \exp_p(tv)$, $t \in [0,1]$ has unique inside U. clearly, $\gamma(0) = p$ and $\gamma(1) = q$. Since $\dot{\rho} = v$, we have $\dot{\gamma}(0) = T\exp_p(\dot{\rho}_0)$

$$= T\exp_p(v)$$

$$= T(\gamma_v(1))$$

$$= T(q) = v$$

under usual identification. in T_pM.

Now, assume that $\gamma_1 : [0,1] \rightarrow U \subseteq M$ is some geodesic with $\gamma_1(0) = p$ and $\gamma_1(1) = q$. If $\dot{\gamma}_1(0) = \omega$, then $\gamma_1(t) = \exp_p(t\omega)$. Now ~~our~~ our claim is that the ray $\rho_1 : t \longmapsto t\omega$ $(t \in [0,1])$ stays inside $\tilde{U}$. If not then the set $A = \{t : t\omega \notin \tilde{U}\}$ (is non-empty) $\neq \phi$. Let $t_* = \inf A$, and let us consider the set $\tilde{C} = \{t\omega : t \in (0, t_*)\}$. Then $\tilde{C} \subseteq \tilde{U}$ and $\tilde{U} \setminus \tilde{C}$ is ~~countable~~ contractible. But its image $\exp_p|_{\tilde{U}}(\tilde{U} \setminus \tilde{C})$

is $U \setminus C$, where C is the image of $(0, t_*)$ under γ_1. certainly $U \setminus C$ is not contractible, which is a contradiction.

Therefore both w & v are in $\tilde{U}$. On the other hand $\exp_p w = \gamma_1(1) = q = \exp_p(v)$.

Since $\exp_p|_{\tilde{U}}$ is a diffeomorphism and hence ~~injective~~ injective, we conclude that $v = w$. Thus by basic uniqueness theorem for geodesic the ~~ge~~ segments γ & γ' are equal & both are given by

$$t \longmapsto \exp_p(tv).$$

** Let (M, g) be a semi-Riemannian manifold of dimension n. Let $p_0 \in M$ and $\{e_0, e_1, \ldots, e_n\}$ be any ~~orthogonal~~ orthonormal basis for the semi-Euclidean scalar product space $(T_{p_0}M, \langle \cdot \rangle_{p_0})$. This basis induces in isometry $I : \mathbb{R}^n_\nu \to T_{p_0}M$ by $x^i \longmapsto \sum x^i e_i$.

If U be a normal nbd centered at $p_0 \in M$, $x_{norm} := I^{-1} \circ (\exp_{p_0}|_{\tilde{U}})^{-1} : U \longrightarrow \mathbb{R}^n_\nu = \mathbb{R}^n$ is a co-ordinate chart with domain U. This co-ordinates are called normal co-ordinate centered at p_0.

THEOREM 5

Let $x_{norm} = (x^1, \ldots, x^n)$ be normal co-ordinates defined on U centered at p_0. Then

$$g_{ij}(p_0) = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle = \varepsilon_i \delta_{ij} \quad \text{for all } i, j.$$

and $\Gamma^i_{jk}(p_0) = 0$ for all i, j.

proof :-

Let $v \in T_{p_0}M$ and $\{e_i\}$ be a autonormal basis and $\{e^i\}$ be the basis of $T^*_{p_0}M$ dual to $\{e_i\}$. Let $v = \sum a^i e_i$. We have $e^i \circ \exp_{p_0}|_{\tilde{U}}^{-1} = x^i$

Now, $\gamma_v(t) = \exp_{p_0}(tv)$ and so, $x^i(\gamma_v(t)) = x^i(\exp_{p_0}(tv))$

$$= e^i(tv)$$

$$= t\, e^i(v) = t a^i$$

Then we have, $v = \dot{\gamma}_v(0) = \sum a^i \frac{\partial}{\partial x^i}\Big|_{p_0}$.

In particular, if $a^i = \delta^i_j$ then $e_i = \frac{\partial}{\partial x^i}\Big|_{p_0}$ and we shall have $\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle_{p_0} = \varepsilon_i \delta_{ij}$.

Since γ_v is a geodesic and $x^i(\gamma_v(t)) = ta^i$, the co-ordinate expression for the geodesic eqn reduces to

$$\Gamma^i_{jk}(\gamma_v(t))\, a^j a^k = 0, \ \forall i$$

In particular, at $p_0 = \gamma_v(0)$; $\Gamma^i_{jk}(\gamma_v(0))\, a^j a^k = 0$

$$\text{ie } \Gamma^i_{jk}(p_0)\, a^j a^k = 0, \ \forall i.$$

But v is arbitrary and hence the n-tuple (a^i) is arbitrary.

Thus the quadratic form defined on $\mathbb{R}^n$ by

$$\varphi^i(u) = \Gamma^i_{jk}(p_i)\, u^j u^k, \quad u = (u^i).$$

is identically zero. Hence the bilinear form

$$\varphi^i : (u, v) \longmapsto \Gamma^i_{jk}(p_i)\, u^j u^k \text{ is identically zero.}$$

This means that $\Gamma^i_{jk}(p_0) = 0$ for all i, j, k.

* Curvature zero, locally flat

Convention: From now on, whenever we write $\exp_p^{-1}$, we must have in mind an open set $\tilde{U}$ and an open set U such that $\exp_p|_{\tilde{U}} : \tilde{U} \to U$ is diffeomorphism. $\exp_p^{-1}$ is an abbriviation for $\exp_p|^{-1}_{\tilde{U}}$ usually $\tilde{U}$ ~~with~~ will be starshaped and thereby U is a normal nbd of the point p.

Definition 16

Let U be a ~~nbd~~ normal nbd of a point p_0 in a semi-Riemannian manifold M. The radius function $r : U \to \mathbb{R}$ is define by

$$r_{p_0}(p) := \| \exp_{p_0}^{-1}(p) \|, \ p \in U.$$

If $(x^1, \ldots, x^n)$ are normal co-ordinates defined on U and centered at p_0, then the radius function is given by

$$r_{p_0} = \left| - \sum_{i=1}^{\nu} (x^i)^2 + \sum_{i=\nu+1}^{n} (x^i)^2 \right|^{1/2}$$

The radius function is smooth except on the set ~~where~~ where it is zero. This zero set is called the local null cone and is the image of the intersection of the null cone in $T_{p_0}M$ with $\tilde{U} = \exp_{p_0}^{-1}(U)$.

The Riemannian space, where the metric is +ve definite, the radius function r is smooth except at the center point p_0. Note that in this case r^2 is smooth event at p_0.

Now suppose that $\gamma : [0,1] \to U \subseteq M$ is the geodesic with $\gamma(0) = p_0$, $\gamma(1) = p$ with $\dot{\gamma}(0) = v$. Then we have a function L defined by, (L is length function)

$$L(\gamma) = r(p).$$

Now let us note that $v = \exp_{p_0}^{-1}(p)$, since $\|\dot{\gamma}\|$ is constant $(\|\dot{\gamma}\| = \|v\|,\ v \in [0,1])$. Then we have

$$\int_0^1 \|\dot{\gamma}\|\, dt = \int_0^1 \|v\|\, dt = \|v\| = r(p) = L(\gamma).$$

More generally, if $r > 0$ such that $\gamma_v : t \mapsto \exp_p(tv)$ is defined for $0 \le t \le r$, then

$$\int_0^r |\langle \dot{\gamma}_v(t), \dot{\gamma}_v(t) \rangle|^{1/2} dt = r \int_0^1 |\langle v, v \rangle|^{1/2} dt$$

$$= r\|v\|.$$

In particular, if v is a unit vector, then the length of the geodesic $\gamma_v\big|_{[0,r]}$ is equal to r.

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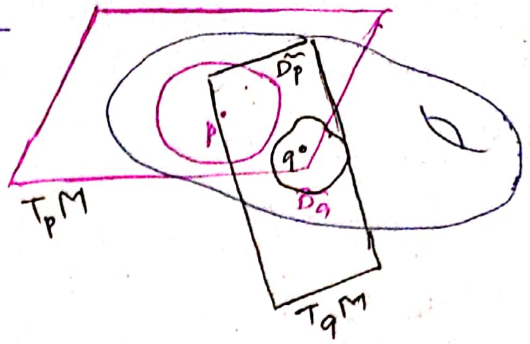
Let $\tilde{D} = \bigcup_{p \in M} \tilde{D}_p$. We can collect the maps $\exp_p : \tilde{D}_p (\subseteq T_pM) \to M$ together to define a map $\exp : \tilde{D} \to M$ given by
$$\exp(v) = \exp_{\pi(v)}(v).$$



The set $\tilde{D}$ is the set of $v \in T_pM$ such that the geodesic γ_v is defined at least on $[0,1]$.

Proposition

$\tilde{D}$ is open and for each $p \in M$, $\tilde{D}_p$ is open and star shaped. Thus $D_p = \exp_p(\tilde{D}_p)$ is a (maximal) normal nbd of p.

proof :-

Let $W \subseteq \mathbb{R} \times TM$ be the domain of geodesic flow $(s, v) \longmapsto \dot{\gamma}_v(s)$. This is the flow of a vector field on TM. and so W is open. W is also the domain of the map $(s, v) \longmapsto \pi \circ \dot{\gamma}_v(s) = \gamma_v(s)$.

Then the map $(1, v) \mapsto v$ is a diffeomorphism from $\{1\} \times TM \to TM$.

Under this diffeomorphism, $\tilde{D}$ corresponds to the set $W \cap \{\{1\} \times TM\}$. And so it must be open in TM and so it.

It also follows that $\tilde{D}_p = \tilde{D} \cap T_pM$ is also open in T_pM and hence D_p is open in M.

To see that $\tilde{D}_p$ is star shaped, let $v \in \tilde{D}_p$. Then γ_v is defined for all $t \in [0,1]$. On the other hand, $\gamma_{tv}(1)$ is defined and equal to $\gamma_v(t)$ for all $t \in [0,1]$. Thus $tv \in \tilde{D}_p$ for all $t \in [0,1]$. Therefore $\tilde{D}_p$ is star shaped.

Let $\Delta = \{(p,p) : p \in M\}$ be the diagonal subset of $M \times M$. Let us defined $Exp : \tilde{D} (\subseteq TM) \longrightarrow M \times M$ by $Exp : v \longmapsto (\pi(v), exp_p v)$, where $\pi : TM \longrightarrow M$ is the tangent bundle projection.

THEOREM

Let O_p is the zero element in T_pM. Then there is a neighbourhood W of O_p in TM s.t. $ExP|_W$ is a diffeomorphism onto a nbd of $(p,p) \in \Delta \subseteq M \times M$.

proof:-

We first show that if $T_x exp_p$ is non singular for some for some $x \in \tilde{D}_p \subseteq T_pM$, then $T_x EXP$ is also non-singular at x. So, let us assume that $T_x exp_p$ is non-singular and suppose that $T_x ExP(v_x) = 0$. We have $\pi = pr_1 \circ ExP$ and so $T\pi(v_x) = Tpr_1(T_x ExP(v_x)) = 0$

This means that v_x is tangent to $\tilde{D}_p \subseteq M$. But the restricted map $ExP|_{\tilde{D}_p}$ is related to exp_p by trivial diffeomorphism:

$$\begin{tikzcd} \tilde{D}_p \arrow[r, "ExP|_{\tilde{D}_p}"] \arrow[d, "id"'] & p \times M \arrow[d, "pr_2"] \\ D_p \arrow[r, leftrightarrow, "exp_p"] & M \end{tikzcd}$$

Thus $T_x exp_p(v_x) = 0$ and hence $v_x = 0$

Since $T_{O_p} exp_p$ is non-singular at each point of the zero section we see that the same is true for $T_{O_p} ExP$. Hence the result follows from inverse ~~map~~ theorem.

Definition 17

A Open subset U of a semi Riemannian manifold is said to be ~~tott~~ totally star shaped if it is a normal nbd of each of its points.

convex normal nbd.

NOTE

U being totally star shaped according to the above definition for any $p, q \in U$, there is a geodesic segment $\gamma : [0,1] \longrightarrow V$ such that $\gamma(0) = p$ & $\gamma(1) = q$ and this is the unique such geodesic with image in U.

THEOREM

Every $p \in M$ has a totally star shaped nbd.

proof :-

Let $p \in M$ and W be a nbd of $O_p \in TM$ s.t. $EXP|_W$ is a diffeomorphism onto a nbd of $(p,p) \in M \times M$. We may assume that $EXP|_W(W)$ is of the form $U(\delta) \times U(\delta)$, where $U(\delta) = \{q : \sum_{i=1}^{n} (x^i(q))^2 < \delta\}$ and $x = (x^1, x^2, \ldots, x^n)$ is a normal co-ordinates system. Let us now consider the tensor b on $U(\delta)$ whose components w.r.to x are $b_{ij} = \delta_{ij} - \Gamma^k_{ij} x^k$. This is clearly symmetric and +ve definite at p and so by choosing δ to sufficiently small we may assume that this tensor is +ve definite on $U(\delta)$.

We shall now show that $U(\delta)$ is a normal nbd of each of its points q. Let $W_q := W \cap T_qM$. We know that $EXP|_{W_q}$ is a diffeomorphism onto $\{q\} \times U(\delta)$ and this means that $\exp_p|_{W_q}$ is a diffeomorphism onto $U(\delta)$. Let us show that W_q is star-shaped about O_q. Let $q' \in U(\delta)$, $q' \neq q$. Let $v = EXP|_{W_q}^{-1} (q, q')$. This means that $\gamma_v : [0,1] \longrightarrow M$ is a geodesic from q to q'. If $\gamma_v([q,1]) \subseteq U(\delta)$, then $tv \in W_q$ for any $t \in [0,1]$ and so, we could conclude that W_q is star-shaped.

Let us assume that $\gamma_v([0,1])$ is not contained in $U(\delta)$. If infact γ_v leaves $U(\delta)$, then the function $f : t \longmapsto \sum_{i=1}^{n} (x^i(\gamma_v(t)))^2$ has a maximum at some $t_0 \in (0,1)$. Thus the second derivative of f cannot be +ve at t_0. We have,

$$\frac{d^2f}{dt^2} = 2\sum_{i=1}^{n}\left(\left(\frac{d}{dt}(x^i\circ\gamma_v)\right)^2 + x^i\circ\gamma_v\,\frac{d^2(x^i\circ\gamma_v)}{dt^2}\right)$$

$$\frac{df}{dx} = x^{i\prime}(\gamma_v).\gamma_v'$$

$$\frac{d^2f}{dt^2} = x^{i\prime\prime}(\gamma_v)(\gamma_v')^2 + x^{i\prime}(\gamma_v)\,\gamma_v''$$

But γ_v is a geodesic and so using geodesic equation we get,

$$\frac{d^2f}{dt^2} = 2\left[\left(\sum_{i,j}\delta_{ij}\frac{d(x^i\circ\gamma_v)}{dt}\frac{d(x^j\circ\gamma_v)}{dt}\right) - \left(\sum_k (x^k\circ\gamma_v)\sum_{i,j}\Gamma_{ij}^k\frac{d(x^i\circ\gamma_v)}{dt}\frac{d(x^j\circ\gamma_v)}{dt}\right)\right]$$

$$= 2\sum_{i,j}\left(\delta_{ij} - \sum_k (x^k\circ\gamma_v)\Gamma_{ij}^k\right)\frac{d(x^i\circ\gamma_v)}{dt}\frac{d(x^j\circ\gamma_v)}{dt}$$

putting $t = t_0$ we have

$$\frac{d^2f(t_0)}{dt^2} = 2b\left(\dot\gamma_v(t_0), \dot\gamma_v(t_0)\right) > 0$$

$$\boxed{g(x^i, x^j) = \sum g_{ij}x^i x^j}$$

which contradicts f having maximum at t_0.

Hence the theorem.

<u>Corallary</u> :- It follows from the proof that given $p\in M$ there is $\delta > 0$ such that $\exp(\{v_p\in T_pM : \|v_p\| < \varepsilon\})$ is totally star shaped for all $\varepsilon < \delta$.

<u>Theorem</u> (Gauss Lemma)

Let $p\in M$, $x\in T_pM$ with $x\neq 0_p$ in the domain of $\exp_p$. Let $v_x, w_x \in T_x(T_pM)$, where v_x, w_x corresponds to $v, w\in T_pM$ under canonical isomorphism between $T_x(T_pM)$ and T_pM. If v_x is radial, i.e., if v is a scalar multiple of x, then

$$\langle T_x\exp_p v_x, T_x\exp_p w_x\rangle = \langle v_x, w_x\rangle.$$

** We shall now introduce the position vector fields associated with a normal nbd of a point p.

First, let us consider a vector space V with scalar product $\langle \cdot , \cdot \rangle$. Then V is a semi-Riemannian manifold with metric defined by $\langle v_x , w_x \rangle := \langle v, w \rangle$. Let ∇ be the associated Levi-Civita connection. Now, $T_v V \cong V$. Then every vector field Y on V can be identified with the map $Y : V \to V$. Under this identification $\nabla_x Y$ is just the directional derivative of Y in the x direction. The position vector field on V is defined by $P : v \mapsto v_v$. In fact its a vector field. Now, $\nabla_X P = X$, for any vector field X. Let us now consider the quadratic form q defined by $q(v) = \langle v, v \rangle$, then $q = \langle P, P \rangle$.

Then we have for any X

$$\begin{aligned} \langle \operatorname{grad} q, X \rangle &= Xq \\ &= X \langle P, P \rangle = \nabla_X \langle P, P \rangle \\ &= 2 \langle \nabla_X P, P \rangle = 2 \langle X, P \rangle \\ &= \langle X, 2P \rangle . \end{aligned}$$

Hence we conclude that $\boxed{\operatorname{grad} q = 2P}$.

It follows that P is normal to every hyper ~~gradr~~ quadratic $q^{-1}(c)$, where $c \in \mathbb{R}$, $c \neq 0$.

$$df = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) (dx, dy, dz).$$

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Unit Radial Vector field

Let us consider the unit sphere S^{n-1} and the map $\mathbb{R}^n \to (0,\infty) \times S^{n-1}$ given by $x \longmapsto (\|x\|, \frac{x}{\|x\|})$. Now let us put co-ordinates on the sphere, say $\theta^1, \theta^2, \dots, \theta^{n-1}$. Composing, we obtained co-ordinates $(r, \theta^1, \theta^2, \dots, \theta^{n-1})$ on an open subset of $\mathbb{R}^n \setminus \{0\}$, where r gives the distance to the origin and the θ-co-ordinates are normal to the r direction. If (M,g) is a Riemannian manifold and if $(r, \theta^1, \dots \theta^{n-1})$ are spherical co-ordinates on $\mathbb{R}^n$ as above, then we can compose normal co-ordinates centred at p to obtained co-ordinate functions on our ~~na~~ normal nbd, which we again denote by $(r, \theta^1, \dots, \theta^{n-1})$. These co-ordinates are called geodesic spherical co-ordinates or geodesic polar-co-ordinates.

$s : \mathbb{R}^n \to (0,\infty) \times S^1$ and $s \circ \exp_p^{-1} : M \longrightarrow (0,\infty) \times S^1$

As usual, the function r is extended to be zero at the center and the angle functions are extended to be multivalued. The resulting "co-ordinates" are not really proper co-ordinates on the normal nbd since they suffer from usual defeats. For example, r is not smooth at the center point where it is zero and the angles $\theta^1, \theta^2, \dots, \theta^{n-1}$ become ambiglaous when $r=0$. ~~Wether~~ weather or not angle functions are introduced. we often use notation $\frac{\partial}{\partial r}$ to denote the unit vector field as follows.

If v is a unit vector in T_pM, then

$$\left.\frac{\partial}{\partial r}\right|_q := \left.\frac{d}{dt}\right|_{t=0} \exp_p(tv), \text{ where } q = \exp_p(tv).$$

(here p is the center point of the normal coordinates).

In fact, it is easy to ~~show~~ see that $\frac{\partial}{\partial r} = \frac{P}{\|P\|}$, where P is the position vector field.

$$P : \mathbb{R}^3 \to \mathbb{R}^3 . \quad P(x,y,z) = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\frac{\partial}{\partial r} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{P}{\|P\|}.$$

We might use this last equation to define $\frac{\partial}{\partial r}$ in the case of an indefinite metric but note that $P/\|P\|$ is undefined when $\|P\|=0$ and so it is undefined on the null cone. We refer to $\frac{\partial}{\partial r}$ as the unit radial vector field. If $(r, \theta^1, \theta^2, \ldots, \theta^{n-1})$ are geodesic spherical co-ordinates centered at some point p_0 of a Riemannian manifold, the by Gauss lemma we have

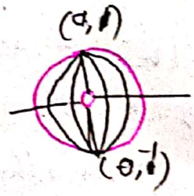
$$\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^i} \right\rangle = 0, \text{ for } i=1,2,\ldots,n-1$$

Riemannian manifolds and distance

Let M be a Riemannian manifold. Let $p, q \in M$. Let path (p,q) be the set of all piecewise smooth curves that begin at p and end at q. We define Riemannian distance from p to q as

$$\text{dist}(p,q) = \inf\{L(c) : c \in \text{path}(p,q)\}$$

On a general Riemannian manifold, $\text{dist}(p,q)=r$ does not necessarily mean that there must be a curve containing p to q having length r. To see this just consider the point $(-1,0)$ and $(1,0)$ on the punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$.



Definition

Let $p \in M$ be a point in a Riemannian manifold M and $R>0$. Then the set $B_R(p)$ (also denoted as $B(p,R)$) is defined by

$$B_R(p) = \{q \in M : \text{dist}(p,q) < R\}$$ and is called an open geodesic ball centered at p with radius R.

Proposition

Let U be a normal nbd of a point p in a Riemannian manifold (M,g). If $q \in U$ and if $\gamma : [0,1] \to M$ is the radial geodesic s.t. $\gamma(0)=p$ and $\gamma(1)=q$, then

γ is the unique shortest curve in U (upto reparametrization) connecting p & q.

Proof:-

Let α be a curve connecting p & q. Without loss of generality we may assume that domain of α be $[0,b]$. Let $\frac{\partial}{\partial r}$ be the radial unit vector field in U. Then we define the vector field R along α by $t \mapsto \frac{\partial}{\partial r}\big|_{\alpha(t)}$. We may then write $\dot{\alpha} = \langle R, \dot{\alpha}\rangle R + N$ for some vector field N normal to R. Then

$$L(\alpha) = \int_0^b \sqrt{\langle \dot{\alpha}, \dot{\alpha}\rangle}\, dt$$

$$= \int_0^b \sqrt{\langle R, \dot{\alpha}\rangle^2 + \langle N, N\rangle}\, dt \geq \int_0^b |\langle R, \dot{\alpha}\rangle|\, dt$$

$$\geq \int_0^b \langle R, \dot{\alpha}\rangle\, dt$$

$$= \int_0^b \frac{d}{dt}(r \circ \alpha)\, dt$$

$$= r(\alpha(b)) = r(q).$$

On the other hand, if $v = \dot{\gamma}(0)$ then $r(q) = \int_0^1 \|v\|\, dt$

$$= \int_0^1 \sqrt{\langle \dot{\gamma}, \dot{\gamma}\rangle}\, dt$$

So, we have $L(\alpha) \geq L(\gamma)$.

Now we shall show that if $L(\alpha) = L(\gamma)$, then α is a reparametrization of γ.

If $L(\alpha) = L(\gamma)$, then all of the above inequalities are exactly equalities, so that N must be identically zero. Then $\frac{d}{dt}(r \circ \alpha) = \langle R, \dot{\alpha}\rangle = |\langle R, \dot{\alpha}\rangle|$. It follows that $\dot{\alpha} = \langle R, \dot{\alpha}\rangle R = \left(\frac{d}{dt}(r \circ \alpha)\right) R$ and so α travels radially from p to q and hence must be a reparametrization of γ.

Proposition

Let p_0 be a point in a Riemannian manifold M. Then there exists a number $\varepsilon_0(p) > 0$ such that for all ε, $0 < \varepsilon \leq (\varepsilon_0)(p)$, we have the following:

1. The open geodesic ball $B(p_0, \varepsilon)$ is normal and has the form $B(p_0, \varepsilon) = \exp_{p_0}\{v \in T_{p_0}M : |v|_g < \varepsilon\}$.

2. For any $p \in B(p_0, \varepsilon)$, the radial geodesic segment connecting p_0 to p is the shortest curve in M, upto reparametrization, ~~for~~ from p_0 to p.

proof :-

Let $\tilde{U} \subseteq T_{p_0}M$ be choosen so that $\exp_p \tilde{U} = 0$ is a normal nbd of p_0. Then for suffiently small $\varepsilon > 0$ the ball

$\tilde{B}(0, \varepsilon) = \{v \in T_pM : \|v\| < \varepsilon\}$ is a star-shaped open set in $\tilde{U}$ and so $A_{p_0, \varepsilon} = \exp_{p_0}(\tilde{B}(0, \varepsilon))$ is a normal nbd of p_0. We know that the radial geodesic segment σ from p_0 to p is the shortest curve in $A_{p_0, \varepsilon}$ from p_0 to p. This curve has length less than ε. We claim that any curve from p_0 to p, whose image ~~in~~ $A_{p_0, \varepsilon}$ must have length greater than ε.

Now, suppose that $\alpha : [a, b] \to M$ curve for p_0 to p leaves $A_{p_0, \varepsilon}$. Then for any $r > 0$ with $r < \varepsilon$, the curve must meet the set

$$s(r) := \exp_{p_0}(\{v \in T_{p_0}M^{-1} : \|v\| = r\})$$ ar

first parameter value $t_1 \in [a, b]$.

Then $\alpha_1 : \alpha|_{[a, t_1]}$ lies in $A_{p_0, \varepsilon}$.

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Then we have $L(\alpha) \geq L(\alpha|_{[a,t_1]}) \geq r$. This is true for any $r < \varepsilon$.

Therefore, we must have $L(\alpha) \geq \varepsilon$.

So, we have

$$A_{p,\varepsilon} = B(p_0, \varepsilon) = \{p \in M : dis(p_0, p) < \varepsilon\}$$

Hence the result.

THEOREM (Distance topology)

Given a Riemannian manifold, let us define the distance function 'dist' as before. Then $(M, dist)$ is a metric space and the metric topology ~~units~~ coincides with the manifold topology of M.

Remark

By definition a ~~co~~ curve segment in a Riemannian manifold say $\gamma : [a,b] \to M$, is a shortest curve if $L(\gamma) = dist(\gamma(a), \gamma(b))$.

We say that such a curve is length minimizing. Such curves must be geodesics.

THEOREM

Let M be a Riemannian manifold. A length minimizing curve $\gamma : [a,b] \to M$ must be a geodesic (unbroken).

Lorentz Vector Space

Def^n

A Lorentz vector space is a scalar product space with index equal to one and dimension greater than or equal to 2.

A Lorentzian manifold is a semi-Riemannian manifold such that every tangent space is a Lorentz vector space with the scalar product given by the metric tensor.

Our conversion is that the signature of a Lorentzian manifold is of the form $(-, +, + \ldots, +)$.

Def^n

The time cone determind by timelike vector v is the set $C^+(v) = \{w \in V : \langle v, w\rangle < 0\}$, V = Lorentz vector space.

Proposition

Let v, w be timelike vectors in a Lorentz vector space. The we have

$|\langle v, w\rangle| \geq \|v\| \; \|w\|$ (Reverse Cauchy Schwatz inequality)

with equality only if v is a scalar multiple of w. Also, if v and w are in the same time cone, then there is a uniquely determined number $\alpha \geq 0$ called the hyperbolic angle between u & w such that

$$\langle v, w\rangle = -\|v\| \|w\| \cosh \alpha.$$

proof:-

We may write $w = av + z$, where $z \in v^\perp$, then

$a^2\langle v, v\rangle + \langle z, z\rangle = \langle w, w\rangle < 0$ [as w is timelik, $\langle v, z\rangle = 0$] .

Now, v is also a timelike vectors. so, $\langle v, v\rangle < 0$.

Then we have, $\langle v, w\rangle^2 = \langle v, av + z\rangle^2$

$$= a^2\langle v, v\rangle^2$$

$$= \frac{\langle w, w\rangle - \langle z, z\rangle}{\langle v, v\rangle} \times \langle v, v\rangle^2$$

$$= \left(\langle w, w\rangle - \langle z, z\rangle\right) \times \langle v, v\rangle$$

$$\geq \langle w, w\rangle \langle v, v\rangle$$

$$\geq \|w\|^2 \|v\|^2$$

Equality holds exactly when $\langle z, z\rangle = 0$. But since $z \in v^{\perp}$, then implies that $z = 0$, so, $w = av$.

Now, if v and w are in same time cone, then we have, $\langle v, w\rangle < 0$

$$\Rightarrow -\langle v, w\rangle \geq \|v\|\,\|w\|$$

$$\Rightarrow \frac{-\langle v, w\rangle}{\|v\|\,\|w\|} \geq 1$$

$|\langle v, w\rangle| \geq \|v\|\,\|w\|$

$-\langle v, w\rangle \geq \|v\|\,\|w\|$,

So, the properties of the function "Cosh" now give a unique number $\alpha \geq 0$ s.t. $\frac{-\langle v, w\rangle}{\|v\| \cdot \|w\|} = \cosh \alpha$.

which implies, $\langle v, w\rangle = -\|v\|\,\|w\| \cosh \alpha$.

Corollary

If v, w are timelike vectors in a Lorentz vectorspace which are in the same time cone, then we have $\|v\| + \|w\| \leq \|v+w\|$. (Reverse triangle inequality). Equality holds if v is a scalar multiple of w.

proof :-

As v, w lie in the same ~~light~~ light cone, $v+w$ is timelike and so $\langle v+w, v+w\rangle < 0$.

$$\therefore \|v+w\|^2 = -\langle v+w, v+w\rangle$$

$$= -\langle v, v\rangle - \langle w, w\rangle - 2\langle v, w\rangle$$

$$\geq \|v\|^2 + \|w\|^2 + 2\|v\| \cdot \|w\|$$

as $\langle v, w\rangle < 0$.

$|\langle v, w\rangle| \geq \|v\|\,\|w\|$

$\Rightarrow -\langle v, w\rangle \geq \|v\|\,\|w\|$

$$\Rightarrow \|v+w\|^2 \geq \left(\|v\| + \|w\|\right)^2$$

$$\Rightarrow \|v+w\| \geq \|v\| + \|w\|$$

Equality holds if $-\langle v, w\rangle = \|v\|\,\|w\|$, which happens only if v is a scalar multiple of w.

Def^n

A Lorentzian manifold (M,g) is said to be time orientable iff there exist a time like vector field $X \in \mathfrak{X}(M)$.

A time orientation of M is a choice of time cone $C^+(p) \subseteq T_pM$ for each $p \in M$ such that there exists a time like $X \in \mathfrak{X}(M)$ with $X_p \in C^+(p)$ for each p.

In particular $C^+(X_p) = C^+(p)$.

$C^+(p)$ is referred to as the future cone. The other time cone at T_pM is called the past time cone.

The timelike vectors in the future time cone are called future pointing and those in the past time cone are called past pointing

future directed.

Lorentzian Manifold

* We mean a spacetime to be a time oriented Lorentzian manifold of dimenssion $n \geq 2$.

* Let (M,g) be a spae time. Let $(p,q) \in M$. we write $p \ll q$ if there is a future directed smooth timelike curve from p to q, and $p \leq q$ if either $p=q$ or there is a smooth future directed causal curve from p to q. Further more we write $p < q$ to mean that $p \leq q$ and $p \neq q$.

* For a given $p \in M$, the chronological future $I^+(p)$, Chronological past $I^-(p)$, causal future $J^+(p)$ and Causal past & $J^-(p)$ are defined as follows

$$I^+(p) = \{q \in M, p \ll q\}$$

$$I^-(p) = \{q \in M, q \ll p\}$$

$$J^+(p) = \{q \in M, p \leq q\}, \quad J^-(p) = \{q \in M : q \leq p\}.$$

* For $A \subseteq M$, we defined

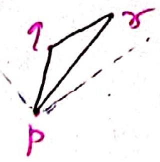
$$I^+(A) = \bigcup_{p \in A} I^+(p) \qquad J^+(A) = \bigcup_{p \in A} J^+(p)$$

$$I^-(A) = \bigcup_{p \in A} I^-(p) \qquad J^-(A) = \bigcup_{p \in A} J^-(p).$$

The relations '$\ll$' and '$\leq$' are clearly transitive.

Moreover, $p \ll q$ and $q < r \Rightarrow p \ll r$.

and $p < q$ and $q \ll r \Rightarrow p \ll r$.



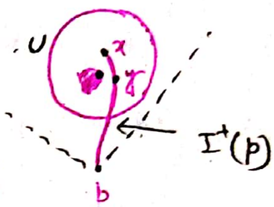
THEOREM

If p be any point in the spacetime (M, g), then $I^+(p)$ and $I^-(p)$ are open subset of M.

proof :-

We just proove that $I^+(p)$ is open and for $I^-(p)$ the result will be ~~analogous~~ analogous.

Let $x \in I^+(p)$. Then there is a smooth future directed curve γ from p to x. Let U be a convex normal nbd of x and let y be a point in U other than x on γ. So now the vector $\exp_y^{-1}(x)$ is time like and future directed (being tangent to γ at y). An so, it belongs to an open set O of time like future directed vectors in $(\exp)_y^{-1}(U)$. Since $\exp_y$ is a diffeomorphism in this nbd, it follows that $\exp_y(O)$ is an open set in U and contains x. Also, $\exp_y(O)$ lies in $I^+(y)$ and therefore in $I^+(p)$. [as '$\ll$' relation is transitive]. Hence $I^+(p)$ is open.



Lemma

For $A \subseteq M$, $I^+(A) = \text{int}(J^+(A))$, $J^+(A) \subseteq \overline{I^+(A)}$

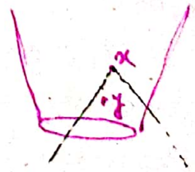
proof:-

Now, $I^+(A) = \bigcup_{x \in A} I^+(x)$ is open, as each $I^+(x)$ is open

The inclusion $I^+(A) \subseteq \text{int}(J^+(A))$ is clear as $I^+(A) \subseteq J^+(A)$ that lines in $J^+(A)$. and $\text{int}(J^+(A))$ is the largest open set

Let $x \in \text{int}(J^+(A))$. Since $\text{int}(J^+(A))$ is open then there is an $y \in I^-(x) \cap \text{int}(J^+(A))$. Hence

$$x \in I^+(y) \subseteq I^+(J^+(A)) = I^+(A)$$



This implies, $\text{int}(J^+(A)) \subseteq I^+(A)$.

Hence $I^+(A) = \text{int}(J^+(A))$.

As, $I^+(A) = \text{int } J^+(A)$

$\Rightarrow \overline{\text{int } J^+(A)} = \overline{I^+(A)}$

$\therefore J^+(A) \subseteq \overline{J^+(A)} \subseteq \overline{\text{int } J^+(A)} = \overline{I^+(A)}$.

$\Rightarrow J^+(A) \subseteq \overline{I^+(A)}$.

* $J^+(p)$ is not necessarily closed :-

Consider the 2 dimensional Minkowski spacetime $\mathbb{R}^2_1 \setminus \{(+1,1)\}$. Let $p = (0,0)$.

Then $J^+(p)$ is not closed as $(2,2)$ is a limit point of $J^+(p)$ but $(2,2) \in J^+(p)$

Def^n

Let U be an open subset of M. We call

$e^+_x(U) = \{y \in M :$ there exists a future directed null geodesic $\gamma \subseteq U$ from x to $y\}$.

the integrated future light cone of x relative to U.

$$C^+_x(U) = \partial I^+(x, U) \rightarrow \text{(Boundary of } I^+(x))$$

C^+_x lives in manifold.

THEOREM

Let (M,g) be a Lorentzian manifold. Then each $x \in M$ has an open nbd U diffeomorphic to $\mathbb{R}^n$ such that $C^+_x(U) \setminus \{x\}$ is a smooth hyper surface which is diffeomorphic to $S^{n-2} \times \mathbb{R}$.

proof :-

Let $u \in T_xM$ be a timelike vector such that $g(u,u) = -1$. Each $v \in T_xM$ can be uniquely decomposed as $v = v^0 u + v'$ where $v' \in u^\perp = \{v \in T_xM : g(u,v) = 0\}$, $v^0 \in \mathbb{R}$. The bilinear form $h(v,w) = g(v,w) + 2v^0 w^0$ (>0) is a Euclidean scalar product on T_xM.

For each $\varepsilon > 0$, let $B_\varepsilon(0_x) = \{v \in T_xM : h(v,v) < \varepsilon\}$. This set is obviously a nbd of 0_x in T_xM. Then $U_\varepsilon = \exp_x(B_\varepsilon(0_x))$ is a convex normal nbd of x for small enough ε. The set

$$S_\varepsilon = \{v \in T_xM : \cancel{g(v,v)=0}\ g(v,v) = 0,\ g(u,v) = -\sqrt{\varepsilon/2}\}$$

is a submanifold of T_xM which is diffeomorphic to

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$S^{n-2} = \{ z \in \mathbb{R}^{n-1} : \sum_{i=1}^{n-2} (z_i)^2 = 1 \}$ and which lies in the boundary of $B_\varepsilon(0_x)$. Now the map $(0,1) \times S_\varepsilon \to M := (t,v) \mapsto \exp_x(tv)$ is a reparametrization of $C_x^+(U_\varepsilon) \setminus \{x\}$. Hence the result.

THEOREM

Let (M,g) be a Lorentzian manifold and U be a convex normal nbd of $x \in M$. Then $y \in I^+(x,U)$ iff $y = \exp_x(v)$ for some future pointing timelike vector v.

proof:-

The exponential map is a diffeomorphism of a nbd $\tilde{U}$ of $0_x \in T_xM$ onto U. For any geodesic γ, we have

$$\nabla_{\dot\gamma}(g(\dot\gamma,\dot\gamma)) = 2g(\nabla_{\dot\gamma}\dot\gamma, \dot\gamma) = 0 \quad , \text{as } \nabla_{\dot\gamma}\dot\gamma = 0.$$

hence the velocity vectors of geodesics do not change their causal class. It follows that $\exp_x$ maps time like vectors into $I^+(x,U)$. We have to show that for each point $y \in I^+(x,U)$, the vector $\exp_x^{-1}(y) = v \in T_xM$ is necessarily time like.

The double cone $\tilde{C}_x = \{ v \in \tilde{U} : g(v,v) = 0 \}$ divides $\tilde{U} \setminus \tilde{C}_x$ into 3 connected components:

The future and past full cones of time like vectors $(\tilde{C}_x^{0+}, \tilde{C}_x^{0-})$ and the set of space like vectors $\tilde{C}_x^{0s}$. Applying the diffeomorphism $\exp_x : \tilde{U} \to U$ we see that the set $C_x(U) = \{ z \in U : \exists v \in \tilde{C}_x \text{ with } z = \exp_x v \}$ divides U into sets $C_x^{0+}, C_x^{0-}, C_x^{0s}$ respectively. Now for any $y \in I^+(x,U)$, there is a smooth timelike curve $\gamma : [a,b] \to U$, $t \mapsto \gamma(t)$ connects x to y. From $g(\dot\gamma(a), \dot\gamma(a)) < 0$, we know that γ must enter C_x^{0+} initially.

If the assertion is not true then $y \in C_x^{0+}(U) \cup C_x^{0s}$. Hence γ must intersect C_x at some point $\gamma(t_0)$ where γ leaves C_x^{0+}. Since $\gamma(t_0)$ is timelike and future directed, it is transverse to C_x at $\gamma(t_0)$ and points into C_x^{0s}. But this is a contradiction to the construction of the point $\gamma(t_0)$.

Hence the result.

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* A spacetime is said to be chronological if $\exists$ no closed timelike curve, i.e. for $x \in M$, $x \notin I^+(x)$.
* A space time (M,g) is said to be causal if $\exists$ no closed causal curve. i.e. for each $x \in M$ $x \notin X$.

Postulate

Let (M,g) be a space time. and $x, y \in M$. x can causaliy influence y iff $y \in J^+(x)$. Material object can reach y from x iff $y \in I^+(x)$.

Def^n

Let $x \in M$, we say that causality (respectively chronology) is violated, iff there exist a closed non-trivial (time like) curve from x to x.

The chronology violated on x is given by $\{x \in M: x \in I^+(x)\}$ and the causality violating set is given by $\{x \in M:$ $\exists$ a non trivial causal curve from x to x.

Lemma

A Lorentzian manifold ~~(M,g)~~ (M,g) is causal (resp. chronological) if the causality ~~vit~~ violated sets (Chronology violated set) is empty.

Def^n

Let (M, g) be a spacetime and $x \in M$. We say that strong causality holds at x if for any nbd U of x there is another nbd ~~of~~ V of x such that $V \subseteq U$ and any causal curve intersects V at most once. i.e. any causal curve intersect V is a connected set.

The space-time (M, g) is said to be strongly causal if it is strongly causal at each point $x \in M$.

Global hyperbolicity

A subset $A \subseteq M$ is said to be globally hyperbolic if A is strongly causal and for any $p, q \in A$ s.t. $J^+(p) \cap J^-(q)$ is compact.

Ex Minkowski spacetime, $\mathbb{R}^4$ is globally hyperbolic

* A subset $A \subseteq M$ is said to be achronal if for any pair of points ~~(x, y)~~ $x, y \in A$, $x \notin I^+(y)$ or $y \notin I^+(x)$.
i.e. if $A \cap I^+(A) = \phi$.

Future set / Past set

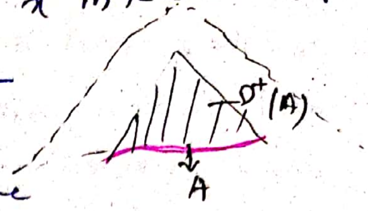
A subset $F \subseteq M$ is said to be a future set if $I^+(F) \subseteq F$.

Anlogously, we can define past set.

A subset $F \subseteq M$ is said to be a past set if $I^-(F) \subseteq F$.

* A curve can not extened further in Future is called Future ~~inexten~~ inextenssible curve.

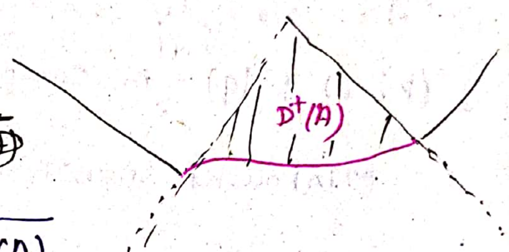
* Let $A \subseteq M$. The future cauchy development $D^+(A)$ is the set of all points $x \in M$ such that all past inextendible causal curve through x intersect A.



Similarly, we can define past cauchy developement $D^-(A)$ and $D(A) := D^+(A) \cup D^-(A)$ is called the cauchy developement of A.

THEOREM

For any achronal set A we have $I^-(\overline{D^+(A)}) \cap I^+(A) \subseteq \overline{D^+(A)}$.



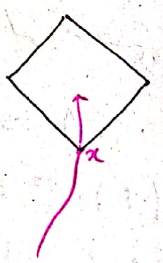
proof :-

~~$x \in I^-(\overline{D^+(A)}) \cap I^+(A)$~~

~~Thus theorem~~

Let $x \in I^-(\overline{D^+(A)}) \cap \overline{I^+(A)}$

Then, there is a point $y \in D^+(A) \cap I^+(x)$. Let γ be a time like curve from x to y. If $x \notin D^+(A)$, Then there is a past inextendible curve μ with future end point x, which does not intersect A.



Since the ~~contradiction~~ concatination of μ and γ in a past inextendible curve with future end point y the curve will intersect A at some point z. This is a contradiction μ does not intersect A.

Hence result.

* In differential geometry, a pseudo-Riemannian manifold (also called semi-Riemannian manifold) is a generalization of a Riemannian manifold in which the metric tensor need not be +ve definite, but is instead only required to be non-degenerate, which is a weaker condition.

* A space time is a semi Rieamannian manifold where g is a non-degenerated metric with signature $(- + + +)$.

Minkowski space (M, η), where η is give by

$$(\eta) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

* Let (M, g) be a Lorentzian manifold and C be a convex normal neighbourhood of $x \in C$. Let $K \subseteq C$ be compact and γ be a causal curve in K. Then show that γ is extendible

$\Rightarrow$ Let $\gamma : [a, b) \to C$ be a future directed causal curve in K. The curve γ can be future extended if ~~the~~ $\lim\limits_{t \to b} \gamma(t)$ exists. In order to see that this limit exist, let $\{\gamma(t_i)\}$; $\{\gamma(s_j)\}$ be convergent sequences with $\lim\limits_{i \to \infty} t_i = \lim\limits_{j \to \infty} s_j = b$ and x, y be their limit points.

For any i there is a $j > i$ with $\gamma(t_j) \in J^+(\gamma(s_i), C)$ and for any j there is an $i > j$ with $\gamma(s_i) \in J^+(\gamma(t_j), C)$. Hence we obtain $x \in J^+(y, C)$ and $y \in J^+(x, C)$.

Hence there are two future directed causal vectors v, w with $x = \exp_y(v)$ and $y = \exp_x(w)$. ~~Traversing~~ Traversing the geodesics $t \mapsto \exp_y(tv)$ backwards we seef that at x there is also a ~~caus~~ causal past directed vector u with $\exp_x(u) = y$. Since the exponential map $\exp_x$ is a diffeomorphism of an open set $\tilde{C} \subseteq T_x M$ to C we must have $w = u$. But this is only possible if ~~on~~ both vectors vanish.

Hence the result.

Chronology violating set of Misner's spacetime $(\mathbb{R}\times S^1, 2\,d\phi\,dt + d\phi^2)$ is given by the set $\{(t, \phi) : t < 0\}$.